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HW5

PHYS 515B
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1. Show that the retarded potential ϕ of a point charge q moving with $x = 0, y = 0, z = vt$ is

$$\phi = \frac{q}{1 - v^2/c^2} \frac{1}{r'}$$

where

$$r' = \left[x^2 + y^2 + \frac{(z - vt)^2}{1 - v^2/c^2}\right]^{1/2}.$$

Solution.

$$\begin{aligned} \rho &= q\delta(\vec{r} - \vec{v}t) \\ &= q\delta(x)\delta(y)\delta(z - vt). \end{aligned}$$

The retarded potential is

$$\begin{aligned} \phi(\vec{r}, t) &= \int \frac{d\vec{r}' dt'}{|\vec{r} - \vec{r}'|} \rho(\vec{r}', t') \delta\left(t - t - \frac{|\vec{r} - \vec{r}'|}{c}\right) \\ &= q \int \frac{dt'}{|\vec{r} - \vec{r}'|} \delta\left(t - t' - \frac{|\vec{r} - \vec{r}'|}{c}\right) \\ |\vec{r} - \vec{r}'| &= \sqrt{x^2 + y^2 + (z - vt')^2}. \end{aligned}$$

Use relationship of delta function

$$\delta(f(t')) = \frac{\delta(t' - t)}{f'(t)}$$

t is the root of $f(t) = 0$.

$$\delta\left(t - t' - \frac{|\vec{r} - \vec{r}'|}{c}\right) = |\vec{r} - \vec{r}'| \frac{\delta(t' - t)}{\sqrt{1 - v^2/c^2}\sqrt{x^2 + y^2 + \frac{(z-vt)^2}{1-v^2/c^2}}}.$$

Therefore,

$$\phi = \frac{q}{\sqrt{1 - v^2/c^2}\sqrt{x^2 + y^2 + \frac{(z-vt)^2}{1-v^2/c^2}}}$$

2. If the acceleration $\mathbf{w}$ of a particle with charge q is perpendicular to the velocity. Show

$$\int d\Omega \frac{d^2W}{dt_R d\Omega} = \frac{2}{3c^3} \frac{\mathbf{w}^2 q^2}{(1 - v^2/c^2)^2}.$$

Solution. Start with the formula from Jackson

$$\frac{dP(t_R)}{d\Omega} = \frac{d^2W}{dt_R d\Omega} = \frac{|\mathbf{n} \times \{(\mathbf{n} - \beta) \times \dot{\beta}\}|^2}{(1 - \mathbf{n} \cdot \beta)^5}.$$

Define the usual polar angles θ, ϕ. WLOG, assume β is in the z-direction and $\dot{\beta}$ in the x-direction. Given this geometry the equation above becomes

$$\frac{d^2W}{dt_R d\Omega} = \frac{q^2}{4\pi c^3} \frac{\mathbf{w}^2}{(1 - \beta\cos\theta)^3} \left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1 - \beta\cos\theta)^2} \right].$$

Integrate over all solid angles Ω, and I get

$$\begin{aligned} \int d\Omega \frac{d^2W}{dt_R d\Omega} &= \frac{q^2\mathbf{w}^2}{4\pi c^3} \int_0^{\pi} \frac{1}{(1 - \beta\cos\theta)^3} \int_0^{2\pi} \left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1 - \beta\cos\theta)^2} \right] d\phi d\theta \\ &= \frac{q^2\mathbf{w}^2}{4c^3} \int_0^{\pi} \left[\frac{2\gamma^2(1 - \beta\cos\theta)^2 - \frac{1}{2}\sin^2\theta}{\gamma^2(1 - \beta\cos\theta)^5} \right] d\theta \\ &= \frac{2}{3c^3} \frac{\mathbf{w}^2 q^2}{(1 - v^2/c^2)^2}. \end{aligned}$$

3. For a particle of charge q moving non-relativistically, if it does a circular motion of radius R in the xy plane with constant angular frequency ω_0, find the time-averaged power radiated per unit angle $dP/d\Omega$.

   Solution. Start with the formula from above, neglecting relativistic effects

$$\frac{dP(t)}{d\Omega} = \frac{2}{3c^3}\mathbf{w}^2 q^2.$$

   Since the system undergoes simple harmonic motion at constant frequency ω_0,

$$\begin{aligned} \mathbf{w}^2 &= \ddot{x}^2 + \ddot{y}^2 \\ &= (-\omega_0^2 x)^2 + (-\omega_0^2 y)^2 \\ &= \omega_0^4 (x^2 + y^2) \\ &= \omega_0^4 R^2 (1 + cos^2\theta). \end{aligned}$$

   This approach assumes $z_0 = 0, \dot{z} = 0$ for all time. Therefore

$$\frac{dP}{d\Omega} = \frac{q^2}{6c^3}\omega_0^4 R^2.$$

4. Jackson 14.21(a)

   Bohr's correspondence principle states that in the limit of large quantum numbers the classical power radiated in the fundamental is equal to the product of the quantum energy ($\hbar\omega_0$) and the reciprocal mean lifetime of the transition from the principal quantum number n to $(n - 1)$.

(a) Using nonrelativistic approximations, show that in a hydrogen-like atom the transition probability (reciprocal mean lifetime) for a transition from a circular orbit of principal quantum number n to $n-1$ is given classically by

$$\frac{1}{\tau} = \frac{2}{3}\frac{e^2}{\hbar c}\left(\frac{Ze^2}{\hbar c}\right)^4 \frac{mc^2}{\hbar}\frac{1}{n^5}.$$

Solution. The problem phrases the power as

$$\begin{aligned} P &= \hbar\omega\frac{1}{\tau} \\ &= \frac{2e^2}{3c^3}|\dot{\mathbf{v}}|^2 \\ &= \frac{2e^2}{3c^3}\omega^4 R^2 \end{aligned}$$

for circular orbits from problem 3. Now, I need to find a relation which gets rid of ω and R.

First, balance forces (centripetal = electrodynamic),

$$\begin{aligned} \frac{mv^2}{R} &= m\omega^2 R = \frac{Ze^2}{R^2} \\ \Longrightarrow R^3 &= \frac{Ze^2}{m\omega^2}. \end{aligned}$$

To get ω in terms of R, use the fact that the angular momentum is quantized, that is

$$\begin{aligned} L &= mvR = n\hbar \\ \Longrightarrow \omega &= \frac{n\hbar}{mR^2}. \end{aligned}$$

Now, isolate the R and ω in terms of quantities in the end result (plug the expression for ω into the expression for R)

$$\begin{aligned} R &= \frac{n^2\hbar^2}{mZe^2} \\ \omega &= \frac{Z^2e^4m}{n^3\hbar^3}. \end{aligned}$$

Plug in these expressions to the acceleration term in the Larmor formula,

$$\begin{aligned} |\dot{\mathbf{v}}|^2 &= (\omega^2 R)^2 \\ &= \frac{m^2Z^6e^{12}}{n^8\hbar^8} \end{aligned}$$

The expression for power becomes,

$$\begin{aligned} P &= \hbar\omega\frac{1}{\tau} \\ &= \frac{2e^2}{3c^3}\frac{m^2Z^6e^{12}}{n^8\hbar^8}. \end{aligned}$$

Divide through by $\hbar\omega$, ω defined from above, to get

$$\begin{aligned}\frac{1}{\tau} &= \frac{2e^2}{3c^3}\frac{mZ^4e^8}{n^5\hbar^6} \\ &= \frac{2}{3}\frac{e^2}{\hbar c}\left(\frac{Ze^2}{\hbar c}\right)^4\frac{mc^2}{\hbar}\frac{1}{n^5},\end{aligned}$$

which is the desired result.

5. At a time $t = 0$, the electron orbits a classical hydrogen atom at a radius a_0 equal to the first Bohr radius. Derive an expression for the time it takes for the radius to decrease to zero due to radiation.

Solution. Take the expression from problem (4) for the power radiated with $n = 1$ and $Z = 1$,

$$P = \frac{2e^2}{3c^3}\frac{m^2e^{12}}{\hbar^8}.$$

The electron will decay to 0 when $E_{\text{radiated}} = P\Delta t$. The initial total energy of the electron is ,

$$E = \frac{1}{2}m\omega^2a_0^2 - \frac{e^2}{2a_0}.$$

Take ω as we had from problem (4), the energy can be rewritten as

$$E = \frac{1}{2}\left(m^2a_0^2\frac{e^4}{\hbar^3} - \frac{e^2}{a_0}\right).$$

Solve for Δt,

$$\begin{aligned}\Delta t &= \frac{E}{P} \\ &= \frac{3c^3\hbar^5}{4e^{10}}\left(a_0^2 - \frac{\hbar^3}{m^2e^2a_0}\right) \\ &\approx 10^{-11}\ \text{sec.}\end{aligned}$$